Fisika Matematika III
Kuliah 7:
Pers. Hermite dan Laguerre

Hasanuddin

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Jurusan Fisika FMIPA UNTAN

Pers. Differensial Hermite

Persamaan Hermite

$$y_n'' - x^2 y_n = -(2n+1)y_n \qquad \dots(1)$$

untuk $n = 0, 1, 2, \dots$

Solusi Hermite dapat dicari dengan metode operator.

Ditulis operator

$$D = \frac{d}{dx}$$

$a^2 - b^2$

$(a-b)(a+b)$

$D^2 y - x^2 y = (D^2 - x^2)\, y$

Metode Operator

$$(D - x)(D + x)y = \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right)y$$

$$= \left(\frac{d}{dx} - x\right)\left(\frac{dy}{dx} + xy\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx} + xy\right) - x\left(\frac{dy}{dx} + xy\right)$$

$$= \frac{d^2y}{dx^2} + y + x\frac{dy}{dx} - x\frac{dy}{dx} - x^2y = y'' - x^2y + y$$

Hubungkan dengan pers. Hermite:

$$(D - x)(D + x)y_n = -2n\, y_n \qquad \ldots(2)$$

Metode Operator

$$(D+x)(D-x)y = \left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right)y$$

$$= \left(\frac{d}{dx}+x\right)\left(\frac{dy}{dx}-xy\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}-xy\right)+x\left(\frac{dy}{dx}-xy\right)$$

$$= \frac{d^2y}{dx^2}-y-x\frac{dy}{dx}+x\frac{dy}{dx}-x^2y = y''-x^2y-y$$

Hubungkan dengan pers. Hermite:

$$(D+x)(D-x)y_n = -2(n+1)y_n \qquad \ldots (3)$$

Resume

$$(D-x)(D+x)y_n = -2n\, y_n \qquad \ldots (2)$$
$$(D+x)(D-x)y_n = -2(n+1)y_n \qquad \ldots (3)$$

Sekarang, ganti y_n di ruas kiri pers. (3) dengan $(D+x)y_m$:

$$(D+x)(D-x)(D+x)y_m = (D+x)(-2m)y_m$$
$$= -2m\,(D+x)y_m \qquad \ldots (4)$$

Kemudian, ganti y_n di ruas kiri pers. (2) dengan $(D-x)y_m$:

$$(D-x)(D+x)(D-x)y_m = (D-x)\big(-2(m+1)\big)y_m$$
$$= -2(m+1)(D-x)y_m \qquad \ldots (5)$$

Amati Pers. (4) dan (5)

$$(D+x)(D-x)[(D+x)y_m] = -2m[\,(D+x)y_m]$$
$$(D-x)(D+x)[(D-x)y_m] = -2(m+1)[(D-x)y_m]$$

Bandingkan dengan

$$(D+x)(D-x)y_n = -2(n+1)y_n \qquad \ldots(3)$$
$$(D-x)(D+x)y_n = -2n\,y_n \qquad \ldots(2)$$

Kita dapat lihat

$$y_n = (D+x)y_m \;\rightarrow n+1 = m \;\rightarrow n = m-1$$
$$y_n = (D-x)y_m \qquad \rightarrow n = m+1$$

Jadi,

$$y_{m-1} = (D+x)y_m \quad \rightarrow (D+x) \;\rightarrow \text{lowering operator}$$
$$y_{m+1} = (D-x)y_m \quad \rightarrow (D-x) \quad \rightarrow \text{raising operator}$$

$n = 0$

Pers. Hermite:

$$(D - x)(D + x)y_n = -2n\, y_n \qquad \dots (2)$$

Untuk $n = 0$:

$$(D - x)(D + x)y_0 = 0$$

$$(D + x)y_0 = 0$$

$$Dy_0 + xy_0 = 0$$

$$\frac{dy_0}{dx} = -xy_0$$

$$\frac{dy_0}{y_0} = -x\, dx$$

$$\int \frac{dy_0}{y_0} = -\int xdx$$

$$\ln y_0 = -\frac{x^2}{2} + C$$

$$y_0 = Ae^{-x^2/2}$$

Fungsi Hermite

Aplikasikan operator $(D - x)$ sebanyak n kali terhadap y_0 untuk mendapatkan y_n:

$$y_n = (D - x)^n y_0 = (D - x)^n e^{-x^2/2}$$

Atau

$$y_n = e^{x^2/2}\, D^n(e^{-x^2})$$

Beberapa fungsi Hemite:

$$y_0 = e^{-x^2/2}$$

$$y_1 = -2x\, e^{-x^2/2}$$

$$y_2 = 2(2x^2 - 1)e^{-x^2/2}$$

$$\ldots$$

Proof

Buktikan $y_n = e^{x^2/2} D^n(e^{-x^2})$

$$e^{x^2/2}D\left(e^{-x^2/2}f(x)\right) = e^{x^2/2}\left[-xe^{-x^2/2}f(x) + e^{-x^2/2}Df(x)\right]$$

$$= e^{x^2/2}e^{-x^2/2}[D-x]f(x) = (D-x)f(x)$$

Substitusikan $f(x)$:

$$f(x) = (D-x)g(x)$$

$$(D-x)f(x) = \boldsymbol{(D-x)^2 g(x)} = e^{x^2/2}D\left(e^{-x^2/2}f(x)\right)$$

$$= e^{x^2/2}D\left(e^{-x^2/2}(D-x)g(x)\right)$$

$$= e^{x^2/2}D\left(e^{-x^2/2}e^{x^2/2}D\left(e^{-x^2/2}g(x)\right)\right) = \boldsymbol{e^{x^2/2}D^2\left(e^{-x^2/2}g(x)\right)}$$

Lanjutkan sampai ke-n:

$$\boldsymbol{(D-x)^n g(x) = e^{x^2/2}D^n\left(e^{-x^2/2}g(x)\right)}$$

Sekarang,

$$y_n = (D-x)^n e^{-x^2/2} = e^{x^2/2}D^n\left(e^{-x^2}\right)$$

Polinomial Hermite

Jika fungsi Hermite dikalikan dengan $(-1)^n\, e^{x^2/2}$ diperoleh polinomial Hermite:

$$(-1)^n\, e^{x^2/2} y_n = (-1)^n\, e^{x^2/2} e^{x^2/2}\, D^n\left(e^{-x^2}\right)$$

$$H_n(x) = (-1)^n\, e^{x^2}\, D^n\left(e^{-x^2}\right)$$

Beberapa polinomial Hermite:

$$H_0(x) = 1$$

$$H_1(x) = -2x$$

$$H_2(x) = 4x^2 - 2$$

dst.

Pers. Diff. untuk Polinomial Hermite

$$y_n = e^{-x^2/2} H_n(x)$$

$$y_n' = -xe^{-x^2/2} H_n + e^{-x^2/2}\, H_n' = (H_n' - xH_n)e^{-x^2/2}$$

$$y_n'' = (H_n'' - H_n - xH_n')e^{-x^2/2} - x(H_n' - xH_n)e^{-x^2/2}$$

$$= (H_n'' - 2xH_n' + (x^2 - 1)H_n)e^{-x^2/2}$$

Substitusikan ke pers. Diff. Hermite

$$y_n'' - x^2 y_n + (2n+1)y_n = 0$$

$$(H_n'' - 2xH_n' + (x^2 - 1)H_n)e^{-x^2/2} - x^2 H_n e^{-x^2/2}$$

$$+ (2n+1)H_n e^{-x^2/2} = 0$$

$$(H_n'' - 2xH_n' + 2nH_n)e^{-x^2/2} = 0$$

$$H_n'' - 2xH_n' + 2nH_n = 0$$

Ortonormalitas Fungsi Hermite

$$\int_{-\infty}^{\infty} y_m y_n \, dx = \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx$$

$$= \begin{cases} 0 & , \text{untuk } m \neq n \\ \sqrt{\pi} 2^n n! & , \text{untuk } m = n \end{cases}$$

Kita akan buktikan ortogonal fungsi Hermite:

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 0 \text{ untuk } m \neq n.$$

Proof of Ortogonality

Tulis pers. Untuk polinomial Hermite:

$$H_n'' - 2xH_n' + 2nH_n = 0$$

$$e^{x^2} D\left(H_n' e^{-x^2}\right) + 2n\, H_n = 0 \quad \times H_m$$

$$e^{x^2} D\left(H_m' e^{-x^2}\right) + 2m\, H_m = 0 \quad \times H_n$$

$$e^{x^2} H_m D\left(H_n' e^{-x^2}\right) + 2n\, H_n H_m = 0$$

$$e^{x^2} H_n D\left(H_m' e^{-x^2}\right) + 2m\, H_m H_n = 0$$

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$$e^{x^2}\left[H_m D\left(H_n' e^{-x^2}\right) - H_n D\left(H_m' e^{-x^2}\right)\right] = 2(n-m) H_m H_n$$

Proof of Ortogonality (2)

$$e^{x^2}\left[H_m D\left(H_n' e^{-x^2}\right) - H_n D\left(H_m' e^{-x^2}\right)\right] = 2(n-m)H_m H_n$$

$$e^{x^2} D\left[(H_m H_n' - H_n H_m{}')e^{-x^2}\right] = 2(n-m)H_m H_n \quad \times e^{-x^2}$$

$$\frac{d}{dx}\left[(H_m H_n' - H_n H_m{}')e^{-x^2}\right] = 2(n-m)\, e^{-x^2} H_m H_n$$

Integralkan dari $x = -\infty$ sampai $x = \infty$:

$$\left[(H_m H_n' - H_n H_m{}')e^{-x^2}\right]_{-\infty}^{\infty} = 2(n-m) \int_{-\infty}^{\infty} e^{-x^2} H_m H_n dx$$

$$0 = 2(n-m) \int_{-\infty}^{\infty} e^{-x^2} H_m H_n dx$$

Jika $m \neq n$, maka

$$\int_{-\infty}^{\infty} e^{-x^2} H_m H_n dx = 0$$

Generating Function and Recursion Relation

Generating Function

$$\Phi(x,h) = e^{2xh - h^2} = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!}$$

Rumus Rekursi

$$H_n'(x) = 2n\, H_{n-1}(x)$$
$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$$

Pers. Diff. Laguerre

$$xy'' + (1 - x)y' + py = 0$$

dengan p suatu bilangan bulat.

Untuk mencari solusi gunakan solusi deret:

Asumsi:

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$y' = \sum_{n=0}^{\infty} (n + s) a_n x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + s)(n + s - 1) a_n x^{n+s-2}$$

Solusi Deret

Dengan menggunakan asumsi deret, pers. Laguerre:

$$xy'' + (1 - x)y' + py = 0$$

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$

$$-\sum_{n=0}^{\infty}(n+s)a_n x^{n+s} + \sum_{n=0}^{\infty} pa_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)^2 a_n x^{n+s-1} + \sum_{n=0}^{\infty}(p-n-s)a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty}(n+s)^2 a_n x^{n+s-1} + \sum_{n=1}^{\infty}(p-n-s+1)a_{n-1} x^{n+s-1} = 0$$

Solusi Deret (2)

Untuk $n = 0$:

$$(0 + s)^2 a_0 x^{0+s-1} = 0$$

$$s^2 = 0$$

$$s = 0$$

Untuk $n > 0$:

$$\sum_{n=1}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} (p - n + 1) a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} [n^2 a_n + (p - n + 1) a_{n-1}] x^{n-1} = 0$$

$$n^2 a_n + (p - n + 1) a_{n-1} = 0$$

Solusi Deret (3)

$$n^2 a_n = -(p - n + 1)a_{n-1}$$

$$a_n = -\frac{(p - n + 1)}{n^2}\, a_{n-1}$$

Koefisien bagi $n > 0$:

$$a_1 = -\frac{p}{1^2} a_0 = -p a_0$$

$$a_2 = -\frac{(p - 1)}{2^2} a_1 = \frac{p(p - 1)}{1^2 2^2} a_0$$

$$a_3 = -\frac{(p - 2)}{3^2}\, a_2 = -\frac{p(p - 1)(p - 2)}{1^2 2^2 3^2} a_0$$

$$\dots$$

$$a_n = (-1)^n \frac{p(p - 1)(p - 2) \dots (p - n + 1)}{(n!)^2} a_0$$

Solusi Deret (4)

$$y_p = a_0\left(1 - px + \frac{p(p-1)}{(2!)^{\,2}}x^2 - \frac{p(p-1)(p-2)}{(3!)^2}x^3 + \cdots\right.$$

$$\left. + (-1)^n \frac{p(p-1)(p-2)\ldots(p-n+1)}{(n!)^2}x^n + \ \ldots\right)$$

Jika $p = n$, diperoleh Polinomial Laguerre:

$$L_n = 1 - nx + \frac{n(n-1)}{(2!)^{\,2}}x^2 - \frac{n(n-1)(n-2)}{(3!)^2}x^3 + \cdots$$

$$+ \frac{(-1)^n}{n!}\ x^n$$

Polinomial Laguerre

$$L_0 = 1$$

$$L_1 = 1 - x$$

$$L_2 = 1 - 2x + \frac{2}{2^2}x^2 = 1 - 2x + \frac{1}{2}x^2$$

$$L_3 = 1 - 3x + \frac{3.2}{2^2}x^2 - \frac{3.2.1}{(3!)^2}x^3$$

$$= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

Rumus Rodrigues, Ortonormalitas, dan Generating Function untuk Polinomial Laguerre

$$L_n(x) = \frac{1}{n!}\, e^x\, D^n(x^n e^{-x})$$

Ortogonormalitas fungsi Laguerre:

$$\int_0^\infty e^{-x}\, L_m(x) L_n(x) dx = \delta_{mn}$$

Generating Function:

$$\Phi(x, h) = \frac{e^{-xh/(1-h)}}{1-h} = \sum_{n=0}^{\infty} L_n(x) h^n$$

Rumus Rekursif Bagi Polinomial Laguerre

$$L'_{n+1}(x) - L'_n(x) + L_n(x) = 0$$

$$(n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + n\,L_{n-1}(x) = 0$$

$$x\,L'_n(x) - n\,L_n(x) + n\,L_{n-1}(x) = 0$$

Associated Laguerre Equation

$$xy'' + (k + 1 - x)y' + ny = 0$$

Memiliki solusi

$$y = L_n^k(x) = (-1)^k D^k L_{n+k}(x)$$

Rumus Rodrigues

$$L_n^k(x) = \frac{x^{-k} e^x}{n!} D^n\left(x^{n+k} e^{-x}\right)$$

Rumus Rekursif Bagi Associated Laguerre Polinomial

$$(n+1)L_{n+1}^{k}(x) - (2n+k+1-x)L_{n}^{k}(x) + (n+k)L_{n-1}^{k}(x) = 0$$

$$x\, DL_{n}^{k}(x) - n\, L_{n}^{k}(x) + (n+k)L_{n-1}^{k}(x) = 0$$

Ortogonormalitas Associated Laguerre Polinomial

$$\int_0^\infty x^k\, e^{-x}\, L_n^k(x) L_m^k(x) dx = \begin{cases} 0\,, & \text{jika } m \neq n \\ \dfrac{(n+k)!}{n!}, & \text{jika } m = n \end{cases}$$

$$\int_0^\infty x^{k+1}\, e^{-x}\, [L_n^k(x)]^2\, dx = (2n+k+1)\frac{(n+k)!}{n!}$$