Fisika Matematika III
Kuliah 6:
Aplikasi Fungsi Bessel

Hasanuddin

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23-09-2021

Bandul yang talinya diperpanjang

Pers. Gerak bandul

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

Untuk osilasi kecil

$$\sin\theta \approx \theta$$

Dan persamaan gerak bandul:

$$ml^2\ddot{\theta} + 2ml\dot{\theta}\frac{dl}{dt} + mgl\theta = 0$$

Panjang tali berubah

$$l = l_0 + vt$$

$$\frac{dl}{dt} = v$$

Pers. Gerak Bandul

Substitusi

$$dt = dl/v$$

Persamaan gerak bandul

$$ml^2\ddot{\theta} + 2ml\dot{\theta}\frac{dl}{dt} + mgl\theta = 0$$

$$l^2v^2\frac{d^2\theta}{dl^2} + 2lv^2\frac{d\theta}{dl} + gl\theta = 0$$

$$\frac{d^2\theta}{dl^2} + \frac{2}{l}\frac{d\theta}{dl} + \frac{g}{v^2 l}\theta = 0$$

Pers. Diff. Bessel

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

$$y'' + \frac{y'}{x} + \left(1 - \frac{p^2}{x^2}\right)y = 0$$

Solusinya

J_p dan Y_p => $y_p = A\,J_p + B\,Y_p$

Misalkan ada solusi $y_p = x^a Z_p(bx^c)$. Bagaimana pDB nya?

$$y'' + (1 - 2a)\frac{y'}{x} + \left[(bc\,x^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

Jika $a = 0$, $b = c = 1$:

$$y'' + \frac{y'}{x} + \left(1 - \frac{p^2}{x^2}\right)y = 0$$

Pers. Bessel

Modified Bessel equation:

$$y'' + (1-2a)\frac{y'}{x} + \left[(bc\,x^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

Memiliki solusi

$$y_p(x) = x^a\,Z_p(bx^c)$$

Dengan

Z_p dapat diganti dengan J_p maupun Y_p atau N_p.

$$\frac{d^2\theta}{dl^2} + \frac{2}{l}\frac{d\theta}{dl} + \frac{g}{v^2 l}\theta = 0$$

$$y'' + (1-2a)\frac{y'}{x} + \left[(bc\, x^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

$y \rightarrow \theta$ dan $x \rightarrow l$:

$$\frac{d^2\theta}{dl^2} + \frac{(1-2a)}{l}\frac{d\theta}{dl} + \left[(bc\, l^{c-1})^2 + \frac{a^2 - p^2c^2}{l^2}\right]\theta = 0$$

$$1 - 2a = 2$$

$$a^2 - p^2c^2 = 0$$

$$l^{2(c-1)} = l^{-1} \quad \rightarrow 2(c-1) = -1$$

$$(bc)^2 = g/v^2$$

Solusi

$$1 - 2a = 2 \quad \Rightarrow \; a = -\frac{1}{2}$$

$$2(c - 1) = -1 \;\; \Rightarrow \; c = \frac{1}{2}$$

$$b^2c^2 = \frac{g}{v^2} \;\; \Rightarrow \; b = \frac{2}{v}\sqrt{g}$$

$$a^2 - p^2c^2 = 0 \;\; \Rightarrow \; p = \pm 1$$

Solusi

$$y_p(x) = x^a \, Z_p(bx^c)$$

$$\theta_1 = l^{-1/2} \, Z_1\left(\frac{2\sqrt{g}}{v} \, l^{1/2}\right)$$

Solusi

- Misalkan

$$u = \frac{2\sqrt{g}}{v}\ l^{1/2} \quad \rightarrow \frac{du}{dl} = \frac{\sqrt{g}}{v}\ l^{-1/2} = \frac{2g}{v^2} u^{-1}$$

Solusi umum

$$\theta = Au^{-1}J_1(u) + Bu^{-1}Y_1(u)$$

Turunan θ terhadap u:

$$\frac{d\theta}{du} = -[Au^{-1}J_2(u) + Bu^{-1}Y_2(u)]$$

Yang diperoleh dari

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$$

Konstanta A dan B

- Keadaan awal bandul

$$t = 0 \;\; \rightarrow l = l_0 \;\; \rightarrow u = \frac{2\sqrt{g}}{v}\, {l_0}^{1/2} = u_0$$

$$\theta = \theta_0 \quad \& \;\; \dot{\theta} = 0 \;\; \rightarrow \frac{d\theta}{du} = 0$$

sehingga

$$A\, {u_0}^{-1} J_1(u_0) + B\, {u_0}^{-1} Y_1(u_0) = \theta_0 \;\; \times Y_2(u_0)$$

$$A\, {u_0}^{-1} J_2(u_0) + B {u_0}^{-1}\, Y_2(u_0) = 0 \;\; \times Y_1(u_0)$$

$$\Rightarrow A\, {u_0}^{-1} J_1(u_0) Y_2(u_0) + B\, {u_0}^{-1} Y_1(u_0) Y_2(u_0) = \theta_0 Y_2(u_0)$$

$$\Rightarrow A\, {u_0}^{-1} J_2(u_0) Y_1(u_0) + B {u_0}^{-1}\, Y_2(u_0) Y_1(u_0) = 0$$

$$A\, {u_0}^{-1} [J_1(u_0) Y_2(u_0) - J_2(u_0) Y_1(u_0)] = \theta_0 Y_2(u_0)$$

Konstanta A dan B

$$A = \frac{u_0\theta_0 Y_2(u_0)}{J_1(u_0)Y_2(u_0) - J_2(u_0)Y_1(u_0)}$$

$$A\, {u_0}^{-1} J_1(u_0) + B\, {u_0}^{-1} Y_1(u_0) = \theta_0 \quad \times J_2(u_0)$$
$$A\, {u_0}^{-1} J_2(u_0) + B {u_0}^{-1}\, Y_2(u_0) = 0 \quad \times J_1(u_0)$$
$$\Rightarrow A\, {u_0}^{-1} J_1(u_0) J_2(u_0) + B {u_0}^{-1} J_2(u_0) Y_1(u_0) = \theta_0 J_2(u_0)$$
$$\Rightarrow A\, {u_0}^{-1} J_1(u_0) J_2(u_0) + B {u_0}^{-1} J_1(u_0) Y_2(u_0) = 0$$
$$B {u_0}^{-1}[J_2(u_0)Y_1(u_0) - J_1(u_0)Y_2(u_0)] = \theta_0 J_2(u_0)$$

$$B = \frac{u_0\theta_0 J_2(u_0)}{J_2(u_0)Y_1(u_0) - J_1(u_0)Y_2(u_0)}$$

$$3A + 2B = 7$$
$$2A + B = 4$$

Penyederhanaan A dan B

- Untuk menyederhanakan A dan B gunakan persamaan

$$J_n(x)Y_{n+1}(x) - J_{n+1}(x)Y_n(x) = -\frac{2}{\pi x}$$

Didapatkan

$$A = \frac{u_0\theta_0 Y_2(u_0)}{J_1(u_0)Y_2(u_0) - J_2(u_0)Y_1(u_0)} = -\frac{\pi u_0^2\theta_0 Y_2(u_0)}{2}$$

$$B = \frac{u_0\theta_0 J_2(u_0)}{J_2(u_0)Y_1(u_0) - J_1(u_0)Y_2(u_0)} = \frac{\pi u_0^2\theta_0 J_2(u_0)}{2}$$

Penyederhanaan dapat dilanjutkan jika dipilih

$$u_0 = \frac{2\sqrt{gl_0}}{v}$$

sehingga

$$J_2(u_0) = 0$$

$$J_2(x) = 0$$

$$x = 0,$$

5.1356 2230,

8.4172 4414,

11.6198 4117,

14.7959 5178,

17.9598 1949,

21.1169 9705,

24.2701 1231,

27.4205 7355, ...

Solusi Bandul

$$u_0 = \frac{2\sqrt{gl_0}}{v}$$

Jika $J_2(u_0) = 0$, maka $B = 0$ dan solusi bandul

$$\theta = Au^{-1}J_1(u) = \frac{A}{u_0\sqrt{l/l_0}}\,J_1\left(u_0\sqrt{l/l_0}\right)$$

karena

$$\theta_0 = \frac{A}{u_0}J_1(u_0)$$

maka

$$\theta = \frac{\theta_0}{J_1(u_0)\sqrt{l/l_0}}J_1\left(u_0\sqrt{l/l_0}\right)$$

Perhitungan

$$\theta_0 = 0.15\ \mathrm{rad}$$

$$u_0 = \frac{2\sqrt{gl_0}}{v} = 21.1169\,9705$$

$$l_0 = 0.1\ \mathrm{m}$$

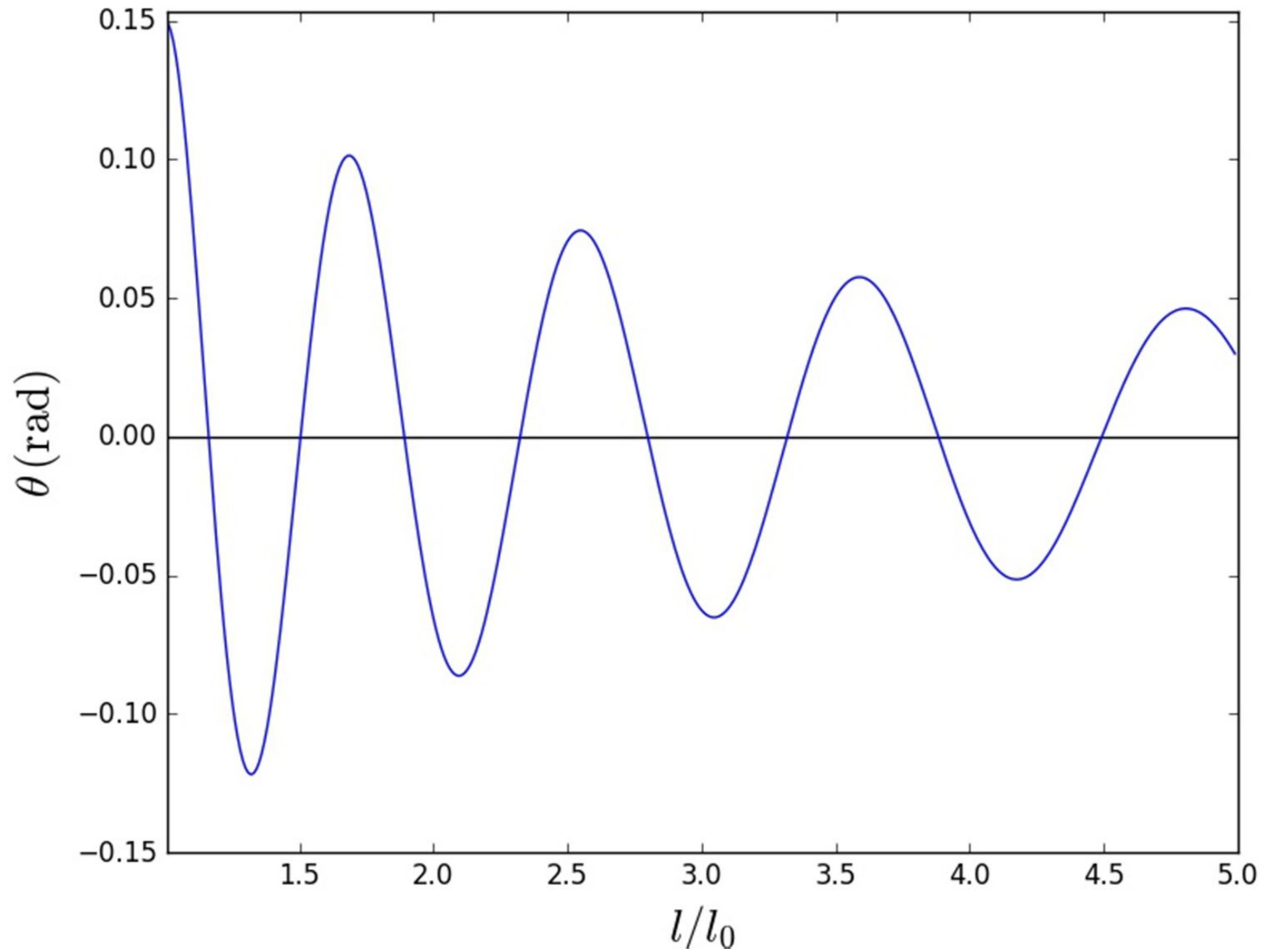
$$g = 9.8\,\mathrm{m/s}^2$$

$$v = \frac{2\sqrt{gl_0}}{u_0} = 0.0937\,5855\,\mathrm{m/s}$$

$$J_1(u_0) = 0.1732647386585233\,8$$

$$\theta = \frac{0.8657272169822481\,9}{\sqrt{l/l_0}} J_1\left(21.1169\,9705\sqrt{l/l_0}\right)$$

Osilasi lengthening pendulum



Kecepatan sudut

$$\frac{d\theta}{du} = -Au^{-1}J_2(u)$$

Nyatakan u dalam l:

$$u = \frac{2\sqrt{gl}}{v} \quad \& \, u_0 = \frac{2\sqrt{gl_0}}{v}$$

Didapat

$$u = u_0\sqrt{l/l_0} \ , \qquad \frac{du}{dl} = \frac{u_0}{2\sqrt{l_0 l}}$$

Karena $A = \theta_0 u_0/J_1(u_0)$:

$$\frac{d\theta}{du} = -\frac{\theta_0 u_0}{J_1(u_0)u_0\sqrt{l/l_0}}J_2\left(u_0\sqrt{l/l_0}\right)$$

Kecepatan sudut dan energi kinetik

$$\frac{d\theta}{dt} = \frac{d\theta}{du}\frac{du}{dl}\frac{dl}{dt}$$

$$= -\frac{\theta_0}{J_1(u_0)\sqrt{l/l_0}} J_2\left(u_0\sqrt{l/l_0}\right)\frac{u_0}{2\sqrt{l_0 l}}v$$

$$= -\frac{\theta_0 u_0 v}{2J_1(u_0)l_0(l/l_0)} J_2\left(u_0\sqrt{l/l_0}\right)$$

$$K = \frac{1}{2}l^2\dot{\theta}^2 = \frac{1}{8}\left[\frac{\theta_0 u_0 v}{J_1(u_0)} J_2\left(u_0\sqrt{l/l_0}\right)\right]^2$$

Energi Potensial

$$U = -gl\cos\theta = -\frac{u^2 v^2}{2}\cos\theta$$

$$= -\frac{u_0^2 v^2}{2}\left(\frac{l}{l_0}\right)\cos\theta$$

Energi Total

$$E = K + U = \frac{1}{2} l^2 \dot{\theta}^2 - \frac{u_0^2 v^2}{2}\left(\frac{l}{l_0}\right)\cos\theta$$