Fisika Matematika III
Kuliah 4: Pers. Diff. Legendre Terasosiasi

Hasanuddin

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Outline

- Deret Legendre
- Fungsi Legendre Terasosiasi
- Metode Frobenius

Deret Legendre

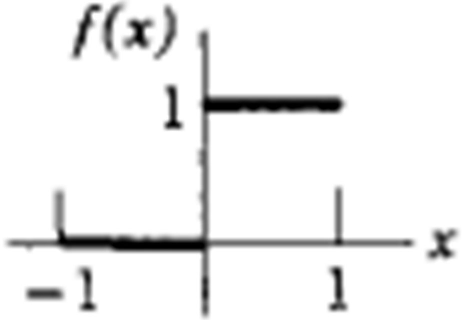
- Karena polinomial Legendre bersifat ortonormal dalam rentang $-1 < x < x$, maka setiap fungsi mungkin dapat diekspansikan dalam deret Legendre.

$$f(x) = \sum_{l=0}^{\infty} c_l \, P_l(x)$$

Contoh:

- Fungsi

$$f(x) = \begin{cases} \mathbf{0}, & -1 < x < 0 \\ \mathbf{1}, & 0 < x < 1 \end{cases}$$



$$f(x) = \sum_{l=0}^{\infty} c_l \, P_l(x)$$

$$\int_{-1}^{1} f(x) \, P_m(x) dx = \int_{-1}^{1} \sum_{l=0}^{\infty} c_l \, P_l(x) \, P_m(x) dx$$

$$= \sum_{l=0}^{\infty} c_l \int_{-1}^{1} P_l(x) P_m(x) dx = c_m \left(\frac{2}{2m+1} \right)$$

$$c_m\left(\frac{2}{2m+1}\right) = \int_{-1}^{1} f(x)\,P_m(x)dx \qquad f(x) = \begin{cases} \mathbf{0}\,, -1 < x < 0 \\ \mathbf{1}\quad, 0 < x < 1 \end{cases}$$

$$c_m = \left(m+\frac{1}{2}\right) \int_{-1}^{1} f(x)P_m(x)\;dx$$

$$c_m = \left(m+\frac{1}{2}\right)\left[\int_{-1}^{0} 0 \times P_m(x)\;dx + \int_{0}^{1} 1 \times P_m(x)\;dx\right]$$

$$c_0 = \frac{1}{2}\int_{0}^{1} dx = \frac{1}{2}$$

$$c_1 = \frac{3}{2}\int_{0}^{1} x\,dx = \frac{3}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$c_2 = \frac{5}{2}\int_{0}^{1}\left(\frac{3}{2}x^2-\frac{1}{2}\right)dx = \frac{5}{2}\left(\frac{1}{2}x^3-\frac{1}{2}x\right)\Bigg|_0^1 = \frac{5}{2}\,(0) = 0$$

$$c_3 = \frac{7}{2}\int_{0}^{1}\frac{1}{2}(5x^3-3x)\,dx = \frac{7}{4}\left(\frac{5}{4}x^4-\frac{3}{2}x^2\right)_0^1 = -\frac{7}{16}$$

Contoh ekspansi fungsi dalam Deret Legendre

$$f(x) = \sum_{l=0}^{\infty} c_l \, P_l(x)$$

$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x)$$

$$+ \frac{11}{32} P_5(x) + \cdots$$

Deret Legendre untuk Pencocokan Kurva

- Misalnya kita ingin mencocokan sembarang fungsi $f(x)$ dengan fungsi polinomial ke-n (contoh $n = 3$)

$$f(x) \cong ax^3 + bx^2 + cx + d$$

Jika kita dapat ekspansikan $f(x)$ dalam deret Legendre

$$f(x) \approx \sum_{l=0}^{3} c_l P_l(x)$$

Pers. Differensial Legendre Terasosiasi

$$(1-x^2)\,y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0$$

dengan $m^2 \leq l^2$.

Solusi dalam bentuk

$$y = (1-x^2)^{m/2}\,u$$

Kita cari u

$$y = (1-x^2)^{\frac{m}{2}} u$$

$$y' = \frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x)u + (1-x^2)^{\frac{m}{2}}u'$$

$$= (1-x^2)^{\frac{m}{2}} u' - mx(1-x^2)^{\frac{m}{2}-1} u$$

$$y'' = (1-x^2)^{\frac{m}{2}} u'' + \frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x)u'$$

$$- m(1-x^2)^{\frac{m}{2}-1} u - mx\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}(-2x)u$$

$$- mx(1-x^2)^{\frac{m}{2}-1} u'$$

$$= (1-x^2)^{\frac{m}{2}} u'' - 2mx(1-x^2)^{\frac{m}{2}-1} u'$$

$$- m(1-x^2-mx^2+2x^2)(1-x^2)^{\frac{m}{2}-2}u =$$

$$= (1-x^2)^{\frac{m}{2}} u'' - 2mx(1-x^2)^{\frac{m}{2}-1} u'$$

$$- m(1+x^2-mx^2)(1-x^2)^{\frac{m}{2}-2}u$$

$(1-x^2)y''$	$(1-x^2)u'' - 2mx\,u' - m(1+x^2-mx^2)(1-x^2)^{-1}u$
$-2xy'$	$-2xu' + 2mx^2(1-x^2)^{-1}\,u$
$l(l+1)y$	$l(l+1)\,u$
$-m^2(1-x^2)^{-1}y$	$-m^2(1-x^2)^{-1}\,u$
Total	$(1-x^2)\,u'' - 2(m+1)x\,u' + l(l+1)u - m(m+1)u$

$$
\begin{aligned}
-m - mx^2 + m^2x^2 + 2mx^2 - m^2 \\
&= m^2x^2 + mx^2 - m^2 - m \\
&= x^2(m^2+m) - 1(m^2+m) \\
&= (x^2-1)(m^2+m) \\
&= -(1-x^2)m(m+1)
\end{aligned}
$$

Persamaan untuk u

$$(1-x^2)\,u'' - 2(m+1)x\,u' + [l(l+1)\; - m(m+1)]\,u = 0 \quad \dots(1)$$

Untuk $m = 0$:

$$u = P_l(x)$$

Turunan persamaan (1)

$$(1-x^2)(u')'' - 2x\,(u')' - 2(m+1)x\,(u')' - 2(m+1)u' + [l(l+1) - m(m+1)]u' = 0$$

$$(1-x^2)(u')'' - 2(m+1+1)x\,(u')' + [l(l+1) - (m+1)(m+2)]u' = 0$$

Solusi:

$$u' = \frac{d}{dx}(P_l)$$

adalah solusi persamaan differensial (1) untuk $m = 1$.

Fungsi Legendre Terasosiasi

$$P_l{}^m = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x)$$

$$P_0{}^0 = P_0(x) = 1$$

$$P_1{}^0 = P_1(x) = x$$

$$P_1^1 = (1-x^2)^{1/2} \frac{d}{dx} P_1 = (1-x^2)^{1/2} \frac{d}{dx} x$$

$$= (1-x^2)^{1/2}$$

$$P_1{}^{-1} \propto P_1{}^1$$

Fungsi Legendre Terasosiasi

Syarat $m^2 \leq l^2$

$l = 0, 1, 2, 3, \ldots \text{(ex. bilangan kuantum orbital)}$

untuk

$$l = 0 \quad \rightarrow m = 0$$

$$l = 1 \ \rightarrow m = -1, 0, 1$$

$$l = 2 \quad \rightarrow m = -2, -1, 0, 1, 2$$

Jumlah bilangan m_l = $2l + 1$.

$$m_l \ \rightarrow \text{bilangan kuantum magnetik}$$

Rumus Rodrigues

$$P_l^m(x) = \frac{1}{2^l\, l!}\,(1-x)^{m/2}\,\frac{d^{l+m}\,(x^2-1)^l}{dx^{l+m}}$$

Sifat Ortonormal

$$\int_{-1}^{1}[P_l^m(x)]^2\,dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$$

Metode Frobenius

- Asumsi solusi PDB

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

- Bagaimana kalau solusinya seperti ini

$$y = \frac{\cos x}{x^2} = \frac{1}{x^2}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right)$$

$$= \frac{1}{x^2} - \frac{1}{2} + \frac{1}{4!}x^2 - \cdots.$$

Atau

$$y = \sqrt{x}\cos x = x^{1/2} - \frac{1}{2}x^{5/2} + \frac{1}{4!}x^{9/2} - \cdots$$

Metode Frobenius

- Asumsi umum

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s}$$

n : bilangan bulat positif

s : bilangan pecahan atau bilangan negatif

Contoh

Tentukan solusi PDB berikut dengan metode deret!

$$x^2y'' + 4xy' + (x^2 + 2)y = 0$$

Asumsi umum:

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$$

$$x^2y'' = \sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s}$$

$$4xy' = \sum_{n=0}^{\infty} 4a_n(n+s)x^{n+s}$$

$$x^2y = \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$2y = \sum_{n=0}^{\infty} 2a_n x^{n+s}$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-1)+4(n+s)+2]x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

$$\sum_{n=0}^{\infty} a_n[n^2 + 2ns + s^2 + 3(n+s) + 2]x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)^2 + 3(n+s) + 2]x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s+1)(n+s+2)]x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0$$

Untuk $n \geq 2$:

$$a_n = -\frac{1}{(n+s+1)(n+s+2)} a_{n-2}$$

Untuk $n = 0$:

$$a_0[(s+1)(s+2)] = 0$$

Karena secara hipotesis $a_0 \neq 0$

$$(s+1)(s+2) = 0$$

$s = -1$ atau $s = -2$.

Ambil $s = -1$.

$$a_n = -\frac{1}{n(n+1)} a_{n-2}$$

$n = 1$

$$a_1[(s+2)(s+3)] = 0 \rightarrow a_1 = 0$$

Pola untuk $s = -1$

$$a_2 = -\frac{1}{2.3} a_0$$

$$a_4 = -\frac{1}{4.5} a_2 = \frac{1}{2.3.4.5} a_0$$

$$a_6 = -\frac{1}{6.7} a_4 = -\frac{1}{2.3.4.5.6.7} a_0$$

$$a_n = \frac{(-1)^{\frac{n}{2}}}{(n+1)!} a_0$$

$$y = \frac{a_0}{x^2}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) = \frac{a_0}{x^2}\sin x$$

Latihan

Tentukan solusi PDB seperti contoh sebelumnya untuk $s = -2$!

Pertanyaan

- Bagaimana cara menentukan Polinomial Legendre ke-l?

Cara untuk menentukan Polinomial Legendre:

Rumus Rekursif

$$P_l(x) = \left(\frac{2l-1}{l}\right) x \, P_{l-1}(x) \; - \left(\frac{l-1}{l}\right) P_{l-2}(x)$$

Rumus Rodrigures

$$P_l(x) = \frac{1}{2^l \, l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$