Fisika Matematika III

Pertemuan ke-3

Generating Function For Legendre Polynomials

Hasanuddin

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Fungsi Pembangkit

Adalah sebuah fungsi yang mewakili suatu deret tak hingga

Contoh:

$$1 + hx + h^2x^2 + h^3x^3 + \ \dots = \sum_{n=0}^{\infty} h^n x^n$$

diwakili oleh fungsi

$$G(x,h) = \frac{1}{1-hx} = 1 + hx + h^2x^2 + h^3x^3 + \ \dots$$

Deret Maclaurin

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$$f(x) = \frac{1}{1-hx} = (1-hx)^{-1} \quad \rightarrow f(0) = 1$$

$$f'(x) = -(1-hx)^{-2}\,(-h) = \frac{h}{(1-hx)^2} \rightarrow f'(0) = h$$

$$f''(x) = -2(1-hx)^{-3}\,(-h)h = \frac{2h^2}{(1-hx)^3} \rightarrow f''(0) = 2h^2$$

$$f(x) = 1 + hx + \frac{2h^2x^2}{2} + \cdots = 1 + hx + h^2x^2 + \ \ldots$$

Fungsi Pembangkit

Tinjau fungsi

$$\Phi(x,h)=\frac{1}{\sqrt{1-2xh+h^2}}=\frac{1}{\sqrt{1-y}}=(1-y)^{-\frac{1}{2}}$$

dengan $y=2xh-h^2$

$$f(y)=(1-y)^{-\frac{1}{2}}\ \Rightarrow f(0)=1$$

$$f'(y)=\frac{1}{2}(1-y)^{-\frac{3}{2}}\ \Rightarrow f'(0)=1/2$$

$$f''(y)=\frac{3}{4}(1-y)^{-\frac{5}{2}}\ \Rightarrow f''(0)=3/4$$

$$f'''(y)=\frac{15}{8}(1-y)^{-\frac{7}{2}}\Rightarrow f'''(0)=15/8$$

…

Deret Mclaurin

$$f(y) = f(0) + f'(0)y + \frac{f''(0)}{2!}\, y^2 + \frac{f'''(0)}{3!} y^3 + \ \ldots$$

$$f(y) = 1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + \cdots$$

Substitusikan $y = 2xh - h^2$ ke deret di atas:

$$\Phi(x,h) = 1 + \frac{1}{2}(2xh - h^2) + \frac{3}{8}(2xh - h^2)^2 + \frac{5}{16}(2xh - h^2)^3 + \ \ldots$$

$$= 1 + xh - \frac{1}{2}h^2 + \frac{3}{8}(4x^2h^2 - 4xh^3 + h^4)$$

$$+ \frac{5}{16}(8x^3h^3 - 4x^2h^4 + \cdots h^5 + \cdots h^6) + \cdots$$

$$= 1 + x\, h + \frac{1}{2}(3x^2 - 1)h^2 + \frac{1}{2}(5x^3 - 3x)h^3 + \cdots$$

$$= P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \ \ldots$$

$$\Phi(x,h) = \sum_{l=0}^{\infty} P_l(x)h^l$$

Fungsi Pembangkit

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}}$$

$$= \sum_{l=0}^{\infty} P_l(x)h^l \quad \text{untuk } |h| < 1.$$

Dengan

$$P_l(x) \rightarrow \text{Polinomial Legendre ke} - l$$

Relasi Rekursif

- Dengan menggunakan fungsi pembangkit, kita dapat merumuskan relasi rekursif.

$$\frac{\partial \Phi}{\partial h} = \Phi_{,h} = \frac{\partial}{\partial h}(1 - 2xh + h^2)^{-\frac{1}{2}} = -\frac{1}{2}(\quad)^{-\frac{3}{2}}(-2x + 2h)$$
$$= \frac{x - h}{(1 - 2xh + h^2)^{3/2}}$$

$$(1 - 2xh + h^2)\Phi_{,h} = (x - h)\Phi$$

$$(1 - 2xh + h^2)\sum_{l=0}^{\infty} l\, P_l(x)h^{l-1} = (x - h)\sum_{l=0}^{\infty} P_l(x)h^l$$

$$\sum_{l=0}^{\infty} l\, P_l(x)h^{l-1} - \sum_{l=0}^{\infty}(2l + 1)xP_l(x)h^l + \sum_{l=0}^{\infty}(l + 1)P_l(x)h^{l+1} = 0$$

$$\sum_{l=0}^{\infty} l\, P_l(x)h^{l-1} - \sum_{l=1}^{\infty}(2l - 1)xP_{l-1}(x)h^{l-1} + \sum_{l=2}^{\infty}(l - 1)P_{l-2}(x)h^{l-1}$$
$$= 0$$

$$\sum_{l=0}^{\infty} l\, P_l(x)h^{l-1} - \sum_{l=1}^{\infty}(2l-1)xP_{l-1}(x)h^{l-1} + \sum_{l=2}^{\infty}(l-1)P_{l-2}(x)h^{l-1}$$

$$1P_1h^0 - xP_0h^0 + \sum_{l=2}^{\infty} l\, P_l(x)h^{l-1} - \sum_{l=2}^{\infty}(2l-1)xP_{l-1}(x)h^{l-1}$$

$$+\sum_{l=2}^{\infty}(l-1)P_{l-2}(x)h^{l-1} = 0$$

$$P_1 - xP_0 = 0$$

$l = 2$:

$$0\, h^0 + \ \dots h^1 + \ \dots h^2 = 0$$

Relasi Rekursif

$l = 0:$

$$0.P_0 = 0 \quad \rightarrow P_0 = 1$$

$l = 1:$

$$P_1 - xP_0 = 0 \ \rightarrow P_1 = xP_0$$

$l \geq 2:$

$$l\,P_l(x) - (2l-1)xP_{l-1}(x) + (l-1)P_{l-2}(x) = 0$$
$$l\,P_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$$

$$P_l(x) = \left(\frac{2l-1}{l}\right)x\,P_{l-1}(x) \ - \left(\frac{l-1}{l}\right)P_{l-2}(x)$$
$$P_2 = \frac{3}{2}xP_1 - \frac{1}{2}P_0 = \frac{3}{2}x^2 - \frac{1}{2}$$
$$P_3 = \frac{5}{3}xP_2 - \frac{2}{3}P_1$$

Ekspansi Potensial

- Potensial oleh muatan titik di suatu titik berjarak d:

$$V = \frac{kq}{d}$$

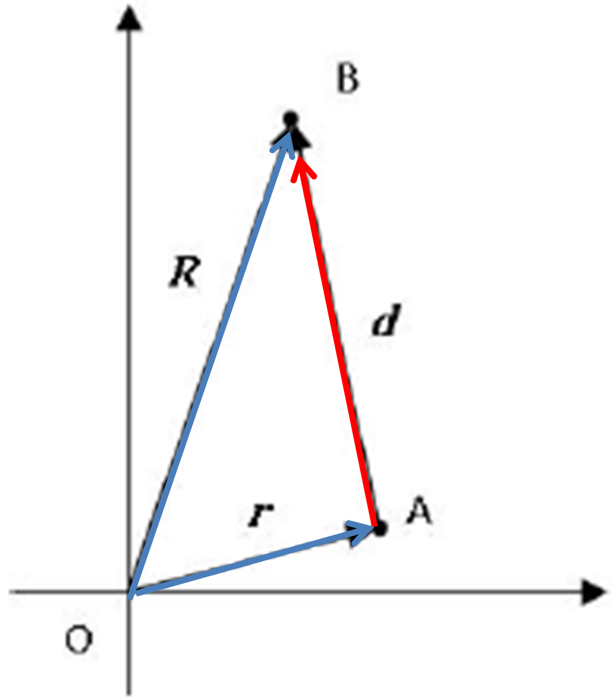
$$d^2 = R^2 + r^2 - 2Rr\cos\theta$$

$$d = \sqrt{R^2 + r^2 - 2Rr\cos\theta}$$

$$V = \frac{kq}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}$$

$$= kq\left(R^2 - 2Rr\cos\theta + r^2\right)^{-1/2}$$

$$= \frac{kq}{R}\left(1 - 2\left(\frac{r}{R}\right)\cos\theta + \left(\frac{r}{R}\right)^2\right)^{-1/2}$$



$$\frac{r}{R} = h$$

$$\cos\theta = x$$

$$= \frac{kq}{R}(1 - 2xh + h^2)^{-1/2}$$

$$V = \frac{kq}{R}\Phi$$

$$V = \frac{kq}{R}\sum_{l=0}^{\infty} P_l(x)h^l = \frac{kq}{R}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{r}{R}\right)^l$$

- Jika ada n muatan titik di sekitar titik O, $\{r_i\}, \{q_i\}$

$$V = \sum_{i=1}^{n} \frac{kq_i}{R} \sum_{l=0}^{\infty} P_l(\cos\theta_i) \left(\frac{r_i}{R}\right)^l$$

$$V = \frac{k}{R} \sum_{i=1}^{n} \sum_{l=0}^{\infty} \left(\frac{r_i}{R}\right)^l P_l(\cos\theta_i) q_i$$

- Jika distribusi muatan kontinu

$$V = \frac{k}{R} \int \sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^l P_l(\cos\theta) dq$$

$$= k \sum_{l=0}^{\infty} \frac{1}{R^{l+1}} \int r^l P_l(\cos\theta) \rho d\tau$$

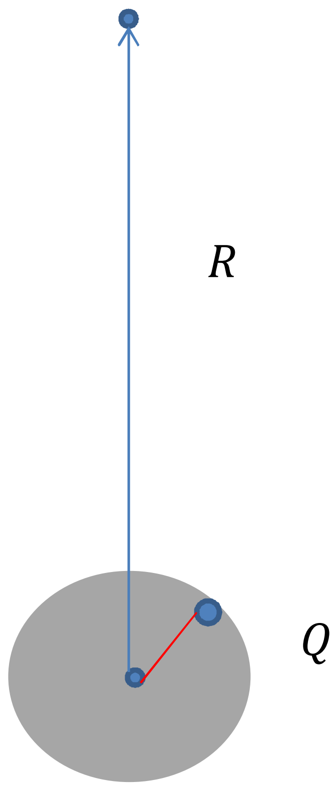
$l = 0:$

$$V_0 = k\frac{1}{R^{0+1}}\int r^0 P_0(\cos\theta)\rho d\tau$$

$$= \frac{kQ}{R}$$

$l = 1:$

$$V_1 = \frac{k}{R^2}\int r\cos\theta\,\rho\,d\tau$$



Ortogonal

- Jika dua buah vektor $\boldsymbol{A}$ dan $\boldsymbol{B}$ saling ortogonal, maka

$$\boldsymbol{A}\cdot\boldsymbol{B} = 0 \ \rightarrow \ \sum A_i B_i$$

- Dua buah fungsi real $f(x)$ dan $g(x)$ ortogonal jika

$$\int f(x)\ g(x)dx \ = 0$$

- Satu set fungsi real dikatakan saling ortogonal jika

$$\int f_m\,(x) f_n(x)\ dx = 0 \ \text{, jika m} \neq n$$

Contoh

$$f_n(x) = \sin nx$$

$$\int_0^{2\pi} \sin mx \sin nx \; dx = 0$$

Polynomial Legendre Saling Ortogonal

$$\int_{-1}^{1} P_n(x)P_l(x)\,dx = 0 \; jika \; n \neq l$$

contoh:

$$n = 0, l = 1$$

$$\int_{-1}^{1} P_0 P_1 dx = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2 = 1 - 1 = 0$$

Sifat Ortonormal

$$\int_{-1}^{1} P_n(x) P_l(x)\, dx = \begin{cases} 0, & n \neq l \\ \dfrac{2}{2l+1}, & n = l \end{cases}$$