Fisika Matematika III

Pertemuan ke-2
Persamaan Legendre

Hasanuddin, Ph.D
26 Agustus 2021

Solusi Latihan Minggu Sebelumnya

PDB

$$xy' = y$$

Solusi:

$$y = a_0x^0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4\,a_4x^3 + \ \ldots$$
$$xy' = 0x^0 + a_1x + 2a_2x^2 + 3a_3x^3 + \cdots$$
$$a_0 = 0$$
$$a_1 = a_1$$
$$a_2 = 2a_2 \ \rightarrow a_2 = 0$$
$$a_3 = 3a_3, \qquad \rightarrow a_3 = 0$$
$$a_n = na_n$$
$$y = a_1x$$

Persamaan Legendre

Bentuk:

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$$

Dengan l adalah suatu tetapan.

Persamaan ini muncul dalam persamaan differensial parsial yang memiliki simetri bola dalam bidang mekanika kuantum, teori elektromagnetika, dan lain-lain.

Solusi

Misalkan

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ \ldots$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

$$= 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \cdots$$

Tabel

	x^0	x	x^2	x^3	x^n
y''	$2a_2$	$2.3.a_3$	$4.3a_4$	$5.4a_5$	$(n+1)(n+2)a_{n+2}$
$-x^2y''$			$-2a_2$	$-2.3a_3$	$-n(n-1)a_n$
$-2xy'$		$-2\ a_1$	$-4a_2$	$-6a_3$	$-2na_n$
$l(l+1)y$	$l(l+1)\ a_0$	$l(l+1)a_1$	$l(l+1)a_2$	$l(l+1)a_3$	$l(l+1)a_n$

$$(n+1)(n+2)a_{n+2} + [l(l+1) - n(n-1) - 2n]a_n = 0$$

$$(n+1)(n+2)a_{n+2} + [l^2 + l - n^2 + n - 2n]a_n = 0$$

$$(n+1)(n+2)a_{n+2} + [l^2 - n^2 + l - n]a_n = 0$$

$$a_{n+2} = -\frac{(l^2 - n^2 + l - n)a_n}{(n+1)(n+2)} = -\frac{(l+n)(l-n) + l - n}{(n+1)(n+2)}a_n$$

$$\boldsymbol{a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+1)(n+2)}a_n}$$

Rumus rekursif

Untuk koefisien pangkat genap

$$a_2 = -\frac{l(l+1)}{1.2} a_0$$

$$a_4 = -\frac{(l-2)(l+3)}{3.4} a_2 = \frac{l(l+1)(l-2)(l+3)}{1.2.3.4} a_0$$

$$a_6 = -\frac{(l-4)(l+5)}{5.6} a_4$$

$$= -\frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{1.2.3.4.5.6} a_0$$

$$a_n = \frac{(-1)^{n/2}}{n!} \left\{ \prod_{k=0}^{n-1} \left(l - (-1)^k k \right) \right\} a_0 \quad , n = 2,4,6, \dots$$

Rumus Rekursif

Untuk koefisien pangkat ganjil

$$a_3 = -\frac{(l-1)(l+2)}{2.3}\,a_1$$

$$a_5 = -\frac{(l-3)(l+4)}{4.5}\,a_3 = \frac{(l-1)(l+2)(l-3)(l+4)}{5!}\,a_1$$

$$a_7 = -\frac{(l-5)(l+6)}{6.7}a_5$$

$$= -\frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{7!}\,a_1$$

$$a_n = \frac{(-1)^{(n-1)/2}}{n!}\left\{\prod_{k=1}^{n-1}\left(l+(-1)^k k\right)\right\}a_1$$

Solusi

$$y = a_0\left\{1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{6!}x^6 + \dots\right\}$$

$$+ a_1\left\{x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 - \frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{7!}x^7 + \dots\right\}$$

Konvergensi

Deret tersebut konvergen jika

$$\lim_{n\to\infty}\left|\frac{a_{n+2}}{a_n}x^2\right| < 1$$
$$|x^2| < 1.$$

Tinjau pada saat

$$l = 0$$

dan $x = 1$,

$$y = a_0 + a_1\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right)$$

Deret dalam tanda kurung divergen. Supaya y konvergen $a_1 = 0$ sehingga

$$y = a_0$$

Konvergensi

- Cek untuk $l = 1$ dan $x = 1$

$$y = a_0 \left\{1 - 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \ \dots \right\} + a_1 x$$

Deret dalam tanda kurung divergen. Supaya konvergen, kita set $a_0 = 0$ dan

$$y = a_1 x$$

- Untuk $l = 2$ dan $x = 1$,

$$y = a_0(1 - 3x^2)$$

dan $a_1 = 0$.

Polinomial Legendre

Solusi:

$$y_0 = a_0 \qquad \rightarrow P_0(x) = 1$$

$$y_1 = a_1 x \qquad \rightarrow P_1(x) = x$$

$$y_2 = a_0(1 - 3x^2) \qquad \rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$y_3 = a_1\left(x - \frac{5}{3}x^3\right) \qquad \rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$y_4 = a_0\left(1 - 10x^2 + \frac{35}{3}x^4\right) \qquad \rightarrow P_4(x) = \frac{1}{8}(3 - 30x^2 + 35x^4)$$

…

Set nilai a_0 sedemikian rupa sehingga

$$P_l(\pm 1) = 1$$

Dan set nilai a_1 sedemikian rupa sehingga

$$P_l(\pm 1) = \pm 1.$$

$$y_2 = a_0(1 - 3x^2)$$

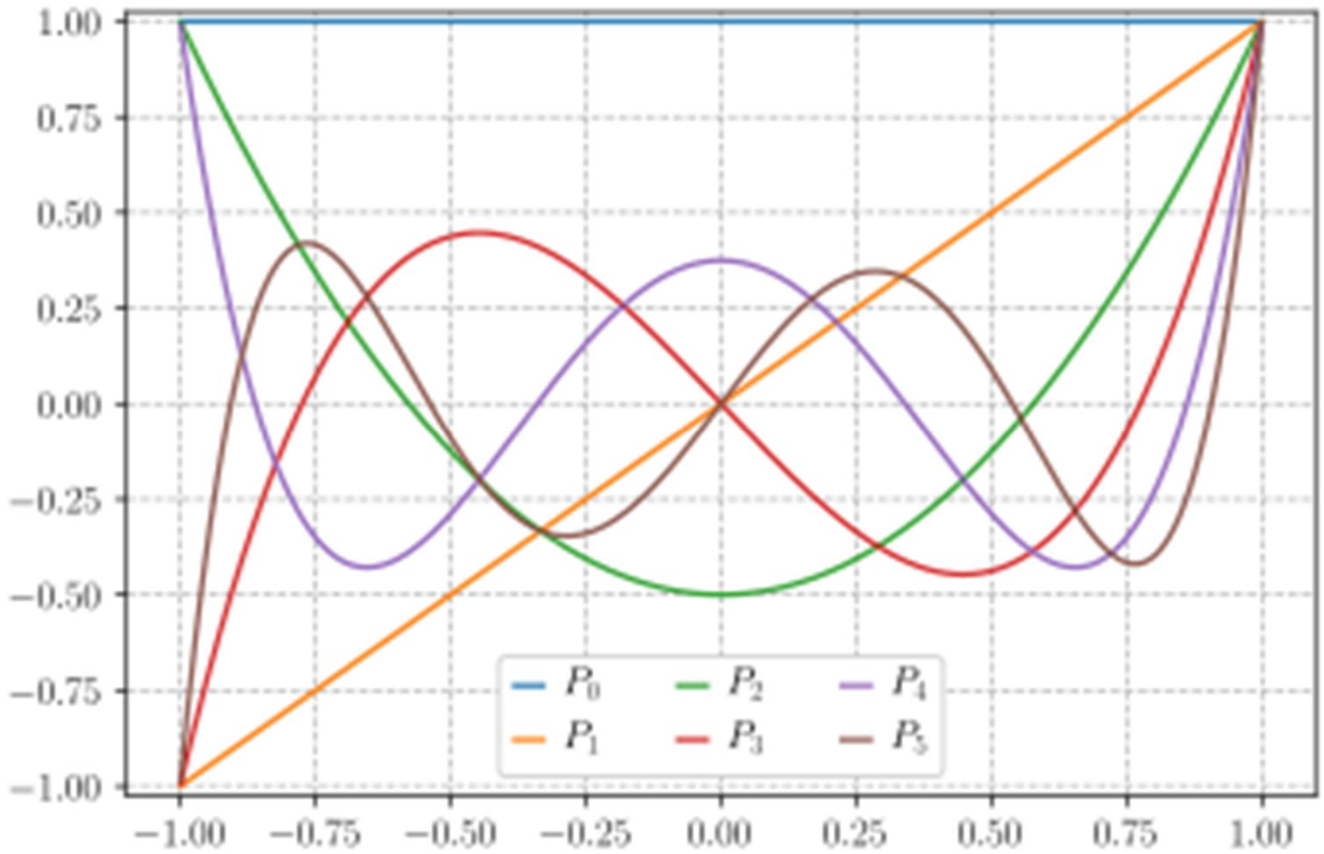
Untuk $x = 1$ -> $y_2 = 1$

$$1 = a_0(1 - 3(1)^2) \Rightarrow 1 = a_0(-2) \Rightarrow a_0 = -\frac{1}{2}$$

$$P_2 = -\frac{1}{2}(1 - 3x^2) = \frac{1}{2}(-1 + 3x^2)$$

$$= \frac{1}{2}(3x^2 - 1)$$

Polinomial Legendre



Nilai dan Fungsi Eigen

Pers. Legendre

$$(x^2 - 1)y'' + 2x\, y' = l(l+1)y$$

$$(x^2 - 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = l(l+1)y$$

$$(x^2 - 1)D^2y + 2x\, Dy = l(l+1)y$$

$$\{(x^2 - 1)D^2 + 2xD\}y = l(l+1)y$$

$$f(D)y = l(l+1)y$$

Nilai eigen l dan fungsi eigen $P_l(x)$.

Aturan Leibniz

$$\frac{d}{dx}(uv) = u'v + uv'$$

$$\frac{d^2}{dx^2}(uv) = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$\frac{d^3}{dx^3}(uv) = u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$$

$$= u'''v + 3\,u''v' + 3u'v'' + uv'''$$

Dst…

Pola segitiga Pascal

1 1

1 2 1

1 3 3 1 -> $(uv)''' = u'''v + 3\,u''v' + 3\,u'v'' + uv'''$

1 4 6 4 1 -> $(uv)'''' = u''''v + 4u'''v' + 6u''v'' + 4u'v''' + u\,v''''$

Aturan Leibniz

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n} C(n,k) \frac{d^{n-k}u}{dx^{n-k}} \frac{d^k v}{dx^k}$$

dengan

$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

Rumus Rodrigues

- Pertanyaan:

Bagaimana mencari polinomial Legendre ke-l?

Tinjau fungsi

$$R = (x^2 - 1)^l$$

Turunan R terhadap x:

$$\frac{dR}{dx} = 2lx(x^2 - 1)^{l-1}$$

$$(x^2 - 1)\frac{dR}{dx} = 2lxR$$

Turunkan persamaan di atas terhadap x:

$$2x\frac{dR}{dx} + (x^2 - 1)\frac{d^2R}{dx^2} = 2lR + 2lx\frac{dR}{dx}$$

$$(1 - x^2)\frac{d^2R}{dx^2} + 2(l - 1)x\,\frac{dR}{dx} + 2lR = 0$$

Apa yang terjadi jika pers. sebelumnya diturunkan sebanyak n kali?

$$(1-x^2)\frac{d^2R}{dx^2}+2(l-1)x\ \frac{dR}{dx}+2lR=0$$

$n=1$

$$-2x\frac{d^2R}{dx^2}+(1-x^2)\frac{d^2}{dx^2}\left(\frac{dR}{dx}\right)+2(l-1)\frac{dR}{dx}$$
$$+2(l-1)x\frac{d^2R}{dx^2}+2l\frac{dR}{dx}=0$$

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{dR}{dx}\right)+2(l-1-1)x\frac{d}{dx}\left(\frac{dR}{dx}\right)$$
$$+[2(l-1)+2l]\frac{dR}{dx}=0$$

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{dR}{dx}\right)+2(l-1-1)x\frac{d}{dx}\left(\frac{dR}{dx}\right)$$
$$+[2(l-1)+2l]\frac{dR}{dx}=0$$

$n=2$

$$-2x\frac{d^2}{dx^2}\left(\frac{dR}{dx}\right)+(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^2R}{dx^2}\right)+2(l-1-1)\frac{d}{dx}\left(\frac{dR}{dx}\right)$$
$$+2(l-1-1)x\frac{d}{dx}\left(\frac{d^2R}{dx^2}\right)+[2(l-1)+2l]\frac{d^2R}{dx^2}=0$$
$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^2R}{dx^2}\right)+2(l-1-2)\,x\frac{d}{dx}\left(\frac{d^2R}{dx^2}\right)$$
$$+[2(l-2)+2(l-1)+2l]\frac{d^2R}{dx^2}=0$$

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^2R}{dx^2}\right)+2(l-1-2)\,x\frac{d}{dx}\left(\frac{d^2R}{dx^2}\right)$$

$$+\,[2(l-2)+2(l-1)+2l]\frac{d^2R}{dx^2}=0$$

$n=3$

$-2x\frac{d}{dx}\left(\frac{d^3R}{dx^3}\right)+(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^3R}{dx^3}\right)+2(l-1-2)\frac{d^3R}{dx^3}+$
$2(l-1-2)\,x\frac{d}{dx}\left(\frac{d^3R}{dx^3}\right)+[2(l-2)+2(l-1)+2l]\ \frac{d^3R}{dx^3}=$
0

Susun ulang:

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^3R}{dx^3}\right)+2(l-1-3)x\frac{d}{dx}\left(\frac{d^3R}{dx^3}\right)$$

$$+\,[2(l-3)+2(l-2)+2(l-1)+2l]\ \frac{d^3R}{dx^3}=0$$

Pola

$$n = 1 \quad \rightarrow 2(l-1-1)x\frac{d}{dx}\left(\frac{dR}{dx}\right) + [2(l-1) + 2l]\frac{dR}{dx}$$

$$n = 2 \ \rightarrow 2(l-1-2)\,x\frac{d}{dx}\left(\frac{d^2R}{dx^2}\right) + [2(l-2) + 2(l-1) + 2l]\frac{d^2R}{dx^2}$$

$$n = 3 \ \rightarrow 2(l-1-3)x\frac{d}{dx}\left(\frac{d^3R}{dx^3}\right) + [2(l-3) + 2(l-2) + 2(l-1) + 2l]\ \frac{d^3R}{dx^3}$$

…

Pola koefisien d^nR/dx^n :

$$2(l-n) + 2(l-n-1) + \cdots + 2(l-1) + 2l = 2l(n+1) - 2\sum_{k=1}^{n} k$$

$$= 2l(n+1) - 2\frac{n(n+1)}{2} = 2l(n+1) - n(n+1) = (2l-n)(n+1)$$

Pers. Legendre

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^nR}{dx^n}\right)+2(l-1-n)\,x\frac{d}{dx}\left(\frac{d^nR}{dx^n}\right)+(2l-n)(n+1)\frac{d^nR}{dx^n}=0$$

Jika $n = l$

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^lR}{dx^l}\right)-2x\frac{d}{dx}\left(\frac{d^lR}{dx^l}\right)+l(l+1)\frac{d^lR}{dx^l}=0$$

Pers. Legendre dengan solusi

$$P_l(x)=k\frac{d^lR}{dx^l}=k\frac{d^l}{dx^l}(x^2-1)^l$$

Rumus Rodrigues untuk Polinomial Legendre

Sebagaimana diketahui

$$P_l(1) = 1$$

$$k\frac{d^l}{dx^l}(x^2-1)^l = 1$$

$$k\frac{d^l}{dx^l}(x+1)^l(x-1)^l = 1$$

$$k\left\{\sum_{m=0}^{l} C(l,m)\frac{d^{l-m}}{dx^{l-m}}(x+1)^l\frac{d^m}{dx^m}(x-1)^l\right\}_{x=1} = 1$$

Suku yang tidak mengandung $(x-1)$ hanya ketika $m = l$:

$$k\{C(l,l)(x+1)^l\, l!\}_{x=1} = 1$$

$$k\left(2^l l!\right) = 1 \quad \rightarrow k = \frac{1}{2^l\, l!}$$

Jadi,

$$P_l(x) = \frac{1}{2^l\, l!}\frac{d^l}{dx^l}(x^2-1)^l$$

Resume

- Solusi Persamaan Differensial Legendre = Polinomial Legendre ($l = bilangan\ bulat$).
- Untuk mencari polinomial Legendre bisa dengan rumus Rodrigues.